6. BABESHKO V.A., GLUSHKOV E.V. and GLUSHKOVA N.V., On the problem of dynamic contact problems in arbitrary domains, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.3, 1978.
7. BABESHKO V.A., GLUSHKOV E.V. and GLUSHKOVA N.V., On singularities at angular points of three-dimensional stamps in contact problems, Dokl. Akad. Nauk SSSR, Vol.257, No.2, 1981.
8. GLUSHKOV E.V. and GLUSHKOV N.V., Plane problem of stamp vibrations on a layer, Izv. Sev.-Kavkaz. Nauch. Tsentra Vysshei Shkoly. Estestv. Nauk, No.1, 1979.
9. VOROVICH I.I., Resonance properties of an elastic inhomogeneous strip, Dokl. Akad. Nauk SSSR, Vol.245, No.5, 1979.

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STABILITY OF ANNULAR PLATES OF INHOMOGENEOUSLY AGEING VISCOELASTIC MATERIAL*

D.M. ZHUKOVITSKII

A thin-plate bending equation in a polar coordinate system is derived for an inhomogeneously ageing material using creep theory. This equation is used to prove the sufficient condition forthe stability of annular plates by an energy method. The case of rigid clamping of both plate edges and compressive forces of dissimilar intensities along these edges is examined. Stresses in the plane of the plate are estimated, whereupon a bound is obtained on the compressible force in explicite form. An extension is made to other kinds of plate support.

Equations for the deflection and sufficient conditions for the stability of inhomogeneously ageing viscoelastic rods were obtained earlier in the one-dimensional case /1/.

1. Formulation of the problem. The strain of an annular plate of constant thickness h and radii R_0 and R $(R_0 < R)$ fabricated from an inhomogeneously ageing viscoelastic material is considered. We introduce a cylindrical system of coordinates $Or\varphi z$ with origin at the centre of the middle plane of the plate in the undeformed state and the Oz axis perpendicular to this plane.

We assume the modulus of instantaneous elastic strain E and Poisson's ratio ν of the plate material to be constant and a load consisting of a transverse distributed load of intensity $q(r, \varphi)$ and compressive forces of intensity p_0 and p on the inner and outer edges of the plate, respectively, to be applied to the plate at the time $t = 0$. We let $\rho(r, \varphi)$ denote the growth of an element of viscoelastic plate material in the neighbourhood of a point with the coordinates r, φ at the time of application of an external load, and L is an operator governing the ageing properties of the material, i.e., /1/

$$Lw(t, r, \varphi) = \int_0^t L(t + \rho(r, \varphi), \tau + \rho(r, \varphi))\, w(\tau, r, \varphi)\, d\tau$$

where $L(t, \tau)$ is the creep kernel. The inverse operator to $I + L$ is denoted by $I - N$: $I - N = (I + L)^{-1}$, where the operator N has the same form as the operator L and governs the relaxation property of the material; the integrand $N(t, \tau)$ is called the relaxation kernel.

Let the following properties of the creep and relaxation kernels be satisfied.

1°. Functions $L_1(t, \tau)$, $N_1(t, \tau)$ exist such that for any $(r, \varphi) \in [R_0, R] \times [0, 2\pi]$, $\tau \in [0, t]$ the inequalities

$$0 \leqslant L(t + \rho(r, \varphi), \tau + \rho(r, \varphi)) \leqslant L_1(t, \tau),\ 0 \leqslant N(t + \rho(r, \varphi), \tau + \rho(r, \varphi)) \leqslant N_1(t, \tau)$$

are satisfied.

2°.
$$|L_1| = \sup_t \int_0^t L_1(t, \tau)\, d\tau < \infty,\quad |N_1| < 1$$

3°. A function $N_0(t, \tau)$ exists for all $\varepsilon > 0$ such that starting at a certain time $t_0 = t_0(\varepsilon) > 0$ for all $t \geqslant \tau \geqslant t_0$

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$$\int_{t_0}^{t} \max_{r,\varphi} |N(t+\rho(\tau, \varphi), \tau+\rho(r, \varphi)) - N_0(t, \tau)| d\tau < \varepsilon$$

We let N_0 denote the operator generated by the function $N_0(t, \tau)$ and L_0 the corresponding operator determined from the relationship $I - N_0 = (I + L_0)^{-1}$.

The Liapunov stability of a plane in an infinite time interval is that small perturbations of the initial plane state result in small values of the deflection $w(t, r, \varphi)$.

Definition. Let $w(t, r, \varphi)$ be the deflection of the plate middle surface, and $q(r, \varphi)$ the transverse load acting on it. The plate is called stable if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if only $\sup_\Omega |q(r, \varphi)| < \delta$, then $\sup_t \max_\Omega |w(t, r, \varphi)| < \varepsilon$ where Ω is the domain of coordinate variation.

We determine the sufficient conditions for stability of the plate under consideration, for which both edges are rigidly clamped. To do this, we first derive an equation for the deflection w. Later we estimate the deflection w by assuming the problem of determining the stresses in the plate to be solved in terms of the compressive forces p_0, p. On the basis of this estimate we obtain the desired conditions. By deriving a constraint on the norm of the stresses in terms of p_0, p and using these conditions we obtain the critical value for the compressive forces.

2. Equation for plate deflection.

Let $\sigma_r, \sigma_\varphi, \sigma_{r\varphi}$ be the stress components, and $\varepsilon_r, \varepsilon_\varphi, \varepsilon_{r\varphi}$ the strain components in the system of polar coordinates. We denote the partial derivatives by the symbol δ_k, where the subscript shows with respect to which variable these derivatives are taken $(k = r, \varphi)$. The rheological relationships for the material described can be obtained in the form /1/

$$\sigma_r = [E/(1-\nu^2)](I-N)(\varepsilon_r + \nu\varepsilon_\varphi), \quad \sigma_{r\varphi} = [E/(2+2\nu)](I - N)\varepsilon_{r\varphi}$$
$$\sigma_\varphi = [E/(1-\nu^2)](I-N)(\varepsilon_\varphi + \nu\varepsilon_r) \tag{2.1}$$

The strains are expressed in terms of the deflection w and the distance z between points of the plate and its middle surface as follows /2, 3/:

$$\varepsilon_r = -z\partial_{rr}{}^2 w, \quad \varepsilon_{r\varphi} = -2z\partial_r(r^{-1}\partial_\varphi w)$$
$$\varepsilon_\varphi = -z(r^{-1}\partial_r w + r^{-2}\partial_{\varphi\varphi}{}^2 w) \tag{2.2}$$

If h is the thickness of the plate, then the bending moments M_r, M_φ and the torque $M_{r\varphi}$ will be

$$M_r = \int_{-h/2}^{h/2} \sigma_r z\,dz, \quad M_\varphi = \int_{-h/2}^{h/2} \sigma_\varphi z\,dz, \quad M_{r\varphi} = \int_{-h/2}^{h/2} \sigma_{r\varphi} z\,dz \tag{2.3}$$

In addition to the transverse forces, compressibe forces will also act on the plate in the case under consideration. The stresses originating here in the plane of the plate will produce an "additional" transverse force. Using the equilibrium equation for a flat plate, we can show /4, 5/ that this force (we denote it by $F(\sigma^\circ, w)$) is described by the expression

$$F(\sigma^\circ, w) = h[\delta_r(\sigma_r{}^\circ\delta_r w) + \delta_r(\sigma_{r\varphi}{}^\circ r^{-1}\delta_\varphi w) + \sigma_r{}^\circ r^{-1}\delta_r w + r^{-1}\delta_\varphi(\sigma_{r\varphi}{}^\circ\delta_r w) + r^{-1}\delta_\varphi(\sigma_\varphi{}^\circ r^{-1}\delta_\varphi w) + \sigma_{r\varphi}{}^\circ r^{-2}\delta_\varphi w] \tag{2.4}$$

We multiply all the relations (2.1) by z and integrate with respect to z between $-h/2$ and $+h/2$. Taking account of (2.2) and (2.3), we obtain

$$M_r = -D(I-N)m_r, \quad m_r = \delta_{rr}{}^2 w + \nu r^{-1}\delta_r w + \nu r^{-2}\delta_{\varphi\varphi}{}^2 w$$
$$M_\varphi = -D(I-N)m_\varphi, \quad m_\varphi = \nu\delta_{rr}{}^2 w + r^{-1}\delta_r w + r^{-2}\delta_{\varphi\varphi}{}^2 w$$
$$M_{r\varphi} = -D(I-N)m_{r\varphi}, \quad m_{r\varphi} = (1-\nu)(r^{-1}\delta_{r\varphi}{}^2 w - r^{-2}\delta_\varphi w)$$
$$D = Eh^3[12(1-\nu^2)]^{-1} \tag{2.5}$$

where D is the cylindrical stiffness of the plate.

The equilibrium equations of an element of a circular plate will be /2/:

for the moments

$$\delta_r(rM_r) + \delta_\varphi M_{r\varphi} - M_\varphi - rQ_r = 0$$
$$\delta_r(rM_{r\varphi}) + \delta_\varphi M_\varphi + M_{r\varphi} - rQ_\varphi = 0$$

for the vertical projections of forces

$$\delta_r(rQ_r) + \delta_\varphi Q_\varphi + r[q(r, \varphi) + F(\sigma^\circ, w)] = 0$$

Eliminating the transverse forces Q_r and Q_φ from these equations, we obtain the equilibrium equation in terms of moments

$$\delta_{rr}{}^2M_r + 2r^{-1}\delta_rM_r + 2r^{-1}\delta_{r\varphi}{}^2M_{r\varphi} - r^{-1}\delta_rM_\varphi + 2r^{-2}\delta_\varphi M_{r\varphi} + r^{-2}\delta_{\varphi\varphi}{}^2M_\varphi + q(r,\varphi) + F(\sigma^\circ, w) = 0$$

We substitute the expression (2.5) here. Extracting components for the identity operator I we obtain

$$\begin{gathered} D\Delta\Delta w + DG(-N, w) = q(r,\varphi) + F(\sigma^\circ, w) \\ G(-N, w) = \delta_{rr}{}^2(-Nm_r) + 2r^{-1}\delta_r(-Nm_r) + 2r^{-1}\delta_{r\varphi}{}^2(-Nm_{r\varphi}) + \\ 2r^{-2}\delta_\varphi(-Nm_{r\varphi}) - r^{-1}\delta_r(-Nm_\varphi) + r^{-2}\delta_{\varphi\varphi}{}^2(-Nm_\varphi) \end{gathered} \tag{2.6}$$

(Δ is the Laplace operator in polar coordinates).

3. Estimate of the deflection $w(t, r, \varphi)$. From the relationship between the creep and relaxation operators we obtain

$$\begin{gathered} -N_0 = (I + L_0)^{-1} - I = -(I + L_0)^{-1}L_0, \\ -N = (N_0 - N) - N_0 = (N_0 - N) - (I + L_0)^{-1}L_0 \end{gathered}$$

Substituting the latter expression for $-N$ into (2.6), we will have

$$D\Delta\Delta w + D(I + L_0)G(N_0 - N, w) = (I + L_0)[q(r,\varphi) + F(\sigma^\circ, w)] \tag{3.1}$$

The rigid clamping of the plate edges will yield boundary conditions for the deflection w as

$$w(t, r, \varphi) = \delta_r w(t, r, \varphi) = 0 \quad \text{when} \quad r = R_0,\ R;\ \forall t \geqslant 0,\ \varphi \in \Omega \tag{3.2}$$

The action of the compressive forces is written analytically in the form

$$\begin{gathered} \sigma_r{}^\circ(t, R_0, \varphi) = -p_0,\ \sigma_r{}^\circ(t, R, \varphi) = -p \\ \sigma_{r\varphi}{}^\circ(t, R_0, \varphi) = \sigma_{r\varphi}{}^\circ(t, R, \varphi) = 0,\ \forall t \geqslant 0,\ \varphi \in \Omega \end{gathered} \tag{3.3}$$

We will assume that an estimate of the stress tensor σ° is known from (2.4) in the norm under condition (3.3) i.e., an estimate of the quantity

$$\|\sigma^\circ\|^2 = \iint(\sigma_r{}^{\circ 2} + 2\sigma_{r\varphi}{}^{\circ 2} + \sigma_\varphi{}^{\circ 2})\,d\omega \tag{3.4}$$

Here and henceforth, the double integral is understood to be integration over the plane of the undeformed plane in polar coordinates, i.e., over r between R_0 and R and over φ between 0 and 2π; $d\omega = r\,dr\,d\varphi$.

We multiply (3.1) by $w(t, r, \varphi)$ and integrate by parts. Because of (3.2) and the continuity and uniqueness in φ for the deflection function $w(t, r, \varphi)$ and all its necessary derivatives, the integrals originating along the boundary vanish. We finally have

$$\begin{gathered} \iint(\Delta w)^2 d\omega = (I + L_0)\iint(N - N_0)(\mathbf{W}, \mathbf{W}_1)\,d\omega + \\ \nu(I + L_0)\iint(N - N_0)(\mathbf{W}, \mathbf{W}_2)\,d\omega + hD^{-1}(I + L_0)\times \\ \iint[(-\sigma_r{}^\circ)\delta_r w\delta_r w_1 + (-\sigma_{r\varphi}{}^\circ)(\delta_r w r^{-1}\delta_\varphi w_1 + r^{-1}\delta_\varphi w\delta_r w_1) + \\ (-\sigma_\varphi{}^\circ)r^{-2}\delta_\varphi w\delta_\varphi w_1]\,d\omega + D^{-1}(I + L_0)\iint wq\,d\omega \\ \mathbf{W} = (\delta_{rr}{}^2 w,\ \sqrt{2}r^{-1}\delta_{r\varphi}{}^2 w - \sqrt{2}r^{-1}\delta_\varphi w,\ r^{-1}\delta_r w + r^{-2}\delta_{\varphi\varphi}{}^2 w) \\ w = w(t, r, \varphi),\ w_1 = w(\tau, r, \varphi) \end{gathered} \tag{3.5}$$

$\mathbf{W}_1 \equiv \mathbf{W}(\tau)$, $\mathbf{W}_2$ is a vector with components of the vector $\mathbf{W}_1$ taken in reverse order and with a minus sign for the second one, and $(\mathbf{W}, \mathbf{W}_i)$ is the scalar product $(i = 1, 2)$.

Let $|\mathbf{W}|$ denote the modulus of the vector, $\|w\|^2$ the integral of w squared with respect to Ω, $\|w\|_i{}^2$ the integral of the sum of the squares of the i-th derivatives $(i = 1, 2)$, and I_j is the j-th component of the right side of (3.5) taken in absolute value $(j = 1, 2, 3, 4)$.

We will estimate the right side of (3.5). To do this we use the following inequalities: Cauchy $|(\mathbf{W}, \mathbf{W}_i)| \leqslant |\mathbf{W}||\mathbf{W}_i|$, Cauchy-Bunyakovskii $|(w, w_1)_2| \leqslant \|w\|_2\|w_1\|_2$ (/6/, p.45 and 135), Bernshtein $\|w\|_2 \leqslant \|\Delta w\|$ (/7/, p.39), Friedrichs-Poincaré $\|w\| \leqslant C(\Omega)\|w\|_1$ (/8/, p.62), written in the polar coordinate system, and also the inequalities

$$\begin{gathered} -2(r^{-1}\delta_{r\varphi}{}^2 w - r^{-2}\delta_\varphi w)(r^{-1}\delta_{r\varphi}{}^2 w_1 - r^{-2}\delta_\varphi w_1) \leqslant \\ \varepsilon(r^{-1}\delta_{r\varphi}{}^2 w - r^{-2}\delta_\varphi w)^2 + \varepsilon^{-1}(r^{-1}\delta_{r\varphi}{}^2 w_1 - r^{-2}\delta_\varphi w_1)^2 \\ \delta_{rr}{}^2 w(r^{-1}\delta_r w_1 + r^{-2}\delta_{\varphi\varphi}{}^2 w_1) \leqslant {}^1/_2\varepsilon(\delta_{rr}{}^2 w)^2 + (2\varepsilon)^{-1}(r^{-1}\delta_r w_1 + \\ r^{-2}\delta_{\varphi\varphi}{}^2 w_1)^2 \\ (r^{-1}\delta_r w + r^{-2}\delta_{\varphi\varphi}{}^2 w)\delta_{rr}{}^2 w_1 \leqslant {}^1/_2\varepsilon(r^{-1}\delta_r w + r^{-2}\delta_{\varphi\varphi}{}^2 w)^2 + \\ (2\varepsilon)^{-1}(\delta_{rr}{}^2 w)^2 \end{gathered}$$

Separating the integral with respect to time into two (from 0 to t_0 and from t_0 to t), and using the properties 1°—3°, we obtain the estimate

$$I_1 \leqslant \| \Delta w \| (1 + | L_0 |) [(| N_1 | + | N_0 |) \| Z_{t_0} \| + \varepsilon_0 \| Z_t \|]$$
$$I_2 \leqslant \varepsilon \| \Delta w \|^2 \, {}^1/_2 \nu (1 + | L_0 |) (| N_1 | + | N_0 |) +$$
$$\nu (2\varepsilon)^{-1} (1 + | L_0 |) [(| N_1 | + | N_0 |) \| Z_{t_0} \|^2 + \varepsilon_0 \| Z_t \|^2]$$
$$Z_t = \sup_\tau | W (\tau, r, \varphi) |$$

Similarly, following the Mikhlink's reasoning (/9/, p.185) we have

$$I_3 \leqslant h | \sigma^\circ |_s (2D\lambda)^{-1} (1 + | L_0 |) \| \Delta w \|^2 + h | \sigma^\circ |_s (2D\lambda)^{-1} (1 + | L_0 |) \| \Delta w_1 \|^2$$

Here

$$\lambda = \inf_w \iint w \Delta\Delta w d\omega \, \{\iint [(\delta_r w)^2 + (r^{-1}\delta_\varphi w)^2]^2 d\omega\}^{-1/2} \quad (3.6)$$
$$| \sigma^\circ |_s = \sup_\tau \| \sigma^\circ \|, \; 0 \leqslant \tau \leqslant t$$

Finally, after double application of the Friedrichs-Poincaré inequality we obtain

$$I_4 \leqslant \varepsilon C (\Omega) (2D) (1 + | L_0 |) \| \Delta w \|^2 + (2D\varepsilon)^{-1} (1 + | L_0 |) \| q \|^2$$

The constant $C(\Omega)$ depends only on the domain over which integration is taken. There is no sense in writing it down explicitly, mainly, it is a bounded constant. Because of ε the value of this same term can be made as small as desired. We discuss below the selection of ε and ε_0 in the estimate I_2.

Substituting the estimates obtained for the integrals I_j into (3.5), we have

$$A \| \Delta w \|^2 - B \| \Delta w \| - C \leqslant 0 \quad (3.7)$$
$$A = 1 - h | \sigma^\circ |_s (2D\lambda)^{-1} (1 + | L_0 |) -$$
$$\varepsilon (1 + | L_0 |) [{}^1/_2 \nu (| N_1 | + | N_0 |) + C (\Omega) (2D)^{-1}],$$
$$B = (1 + | L_0 |) [(| N_1 | + | N_0 |) \| Z_{t_0} | + \varepsilon_0 \| Z_t \|]$$
$$C = h | \sigma^\circ |_s (2D\lambda)^{-1} (I + L_0) \| \Delta w_1 \|^2 + (2D\varepsilon)^{-1} (1 +$$
$$| L_0 |) \| q \|^2 + \nu (2\varepsilon)^{-1} (1 + | L_0 |) [(| N_1 | +$$
$$| N_0 |) \| Z_{t_0} \| + \varepsilon_0 \|Z_t \|]$$

The constants ε and ε_0 are selected in such a manner that the magnitudes and signs of the expressions A, B and C were determined by terms that do not contain ε and ε_0.

Let $h | \sigma^\circ |_s (1 + | L_0 |)/D = \alpha$. If $1 - \alpha /(2\lambda) > 0$, i.e. $A > 0$, then we have from (3.7)

$$\| \Delta w \| \leqslant B/A + (C/A)^{1/2} \quad (3.8)$$

According to the Bernshtein inequality /7/ we obtain $\| Z_t \| \leqslant \sup_\tau \|\Delta w_1 \|^2$ for Z_t. Then taking account of the inequality $(a + b)^{1/2} \leqslant a^{1/2} + b^{1/2}$ for $a, b > 0$, we have

$$B/A \leqslant (1 + | L_0 |) (| N_1 | + | N_0 |)/A \| Z_{t_0} \| + \varepsilon_0 (1 +$$
$$| L_0 |)/A \sup_\tau \| \Delta w_1 \|$$
$$(C/A)^{1/2} \leqslant [\alpha/(2\lambda A) + \varepsilon_0 \nu (1 + | L_0 |)/(2\varepsilon A)]^{1/2} \sup_\tau \| \Delta w_1 \| +$$
$$[\nu (1 + | L_0 |) (| N_1 | + | N_0 |)/(2\varepsilon A)]^{1/2} \| Z_{t_0} \| + [(1 +$$
$$| L_0 |)/(2AD\varepsilon)]^{1/2} \| q \|$$

or introducing the notation

$$B/A \leqslant C_2 \| Z_{t_0} \| + \varepsilon_0 A_2 \sup_\tau \| \Delta w_1 \|$$
$$(C/A)^{1/2} \leqslant A_1 \sup_\tau \| \Delta w_1 \| + C_1 \| Z_{t_0} \| + B_1 \| q \|$$

Taking account of the last inequalities, we write condition (3.8) in the form

$$\| \Delta w \| \leqslant (A_1 + \varepsilon_0 A_2) \sup_\tau \| \Delta w_1 \| + B_1 \| q \| + (C_1 + C_2) \| Z_{t_0} \|$$

Since this inequality is satisfied for all $t \geqslant 0$, we can then obtain

$$(1 - A_1 - \varepsilon_0 A_2) \sup_\tau \| \Delta w_1 \| \leqslant B_1 \| q \| + (C_1 + C_2) \| Z_{t_0} \| \quad (3.9)$$

Theorem. Let conditions 1°—3° be satisfied for problem (2.6), (3.2). Then if $1 - [\alpha/(2\lambda A)]^{1/2} > 0$, it follows that condition (3.9) assures Liapunov stability of an annular plate.

Proof. The $\max | w |$ enters the stability definition while an upper bound is obtained for $\| \Delta w \|$. Therefore, it remains to find the quantity $\| \Delta w \|$ as an upper bound for $\max | w |$. For finite functions we obtain the inequality (/8/, p.84)

$$\max_\Omega | w | \leqslant C (\Omega) \| D^l w \|_{m, \Omega}, \; lm > n \quad (3.10)$$

In the case under consideration, any support on the plate (no free edges) means that $w = 0$ on the plate boundaries, i.e., the deflections w are finite functions. Furthermore, n is the dimensionality of the space and equals two (the plane case), l is the order of differentiation, also equal to two, and integration in all the calculations was performed with the square (the norm was taken in L_2), i.e., $m = 2$ also, therefore, inequality (3.10) is satisfied. Here $C(\Omega)$ is determined over the domain of integration, i.e, over the plate dimensions. Finally, having used the Bernshtein inequality /7/, we obtain $\max_\Omega |w| \leqslant C(\Omega)$, $\| \Delta w \|$. This and (3.9) ensure Liapunov stability of the plate under consideration.

The condition of the theorem determines the constraint on the compressive forces since $\lambda > \alpha$ follows from $1 - [\alpha/(2\lambda A)]^{1/2} > 0$, or

$$\| \sigma^\circ \| \leqslant | \sigma^\circ |_s \leqslant D\lambda [h(1 + | L_0 |)]^{-1} \tag{3.11}$$

Remark. To clarify the meaning of the parameter λ in (3.11), we turn to (3.6). The Cauchy-Bunyakovskii inequality /6/ yields

$$\iint [(\delta_r w)^2 + (r^{-1}\delta_\varphi w)^2]\, d\omega \leqslant \{\pi (R^2 - R_0^2) \iint [(\delta_r w)^2 + (r^{-1}\delta_\varphi w)^2]^2\, d\omega\}^{1/2}$$

By hence deriving the inequality for the inverse quantities, we obtain

$$\lambda \leqslant [\pi (R^2 - R_0^2)]^{1/2} \inf_w \iint w \Delta\Delta w d\omega \, \{\iint [(\delta_r w)^2 + (r^{-1}\delta_\varphi w)^2]\, d\omega\} = \lambda_0 [\pi (R^2 - R_0^2)]^{1/2}$$

λ_0 is the least eigennumber of the corresponding elastic problem. On the other hand, $\lambda > 0$. Indeed, by virtue of /8, p.84/ and the Friedrichs-Poincaré inequality a constant $C_3 > 0$ exists dependent only on the domain Ω such that

$$\{\iint [(\delta_r w)^4 + (r^{-1}\delta_\varphi w)^4]\, d\omega\}^{1/2} \leqslant C_3 \iint (\Delta w)^2\, d\omega$$

This indeed proves that $\lambda > 0$.

4. A priori estimates of the stresses in a ring. If the plate deflection is zero, then a state of stress and strain is realized that is characterizable by the stress tensor σ° from (2.4). The equilibrium equations of a plate element in this case will have the form /2/

$$\delta_r (r\sigma_r^\circ) + \delta_\varphi \sigma_{r\varphi}^\circ - \sigma_\varphi^\circ = 0, \;\; \delta_r (r\sigma_{r\varphi}^\circ) + \delta_\varphi \sigma_\varphi^\circ + \sigma_{r\varphi}^\circ = 0 \tag{4.1}$$

Relationships (3.3) will be their boundary conditions.

We will estimate $\| \sigma^\circ \|$ in terms of p_0 and p. To do this $\| \sigma^\circ \|$ will first be estimated in terms of $\| \varepsilon^\circ \|$, and then $\| \varepsilon^\circ \|$ in terms of p_0 and p. The first estimate is obtained by substituting the stress tensor components

$$\| \sigma^\circ \|^2 = E (1 + \nu)^{-1} J_1 + \nu (1 + \nu)^{-1} \| \sigma^\circ \|_1$$
$$J_1 = \iint [\sigma_r^\circ (I - N)\, \varepsilon_r^\circ + 2\sigma_{r\varphi}^\circ (I - N)\, \varepsilon^\circ_{r\varphi} + \sigma_\varphi^\circ (I - N)\, \varepsilon_\varphi^\circ]\, d\omega,$$
$$\| \sigma^\circ \|_1^2 = \iint (\sigma_r^\circ + \sigma_\varphi^\circ)^2\, d\omega = E (1 - \nu)^{-1} \iint (\sigma_r^\circ + \sigma_\varphi^\circ) (I - N) (\varepsilon_r^\circ + \varepsilon_\varphi^\circ)\, d\omega$$

into $\| \sigma^\circ \|$ of (3.1).

Hence taking account of the inequality

$$(1 - \nu) (1 + \nu)^{-1} \| \sigma^\circ \|^2 \leqslant \| \sigma^\circ \|^2 - \nu (1 + \nu)^{-1} \| \sigma^\circ \|_1^2 = E (1 + \nu)^{-1} J_1$$

we obtain as above

$$| \sigma^\circ |_s \leqslant E (1 - \nu)^{-1} (1 + | N_1 |) \, | \varepsilon^\circ |_s, \;\; | \varepsilon^\circ |_s = \sup_t \| \varepsilon^\circ \| \tag{4.2}$$

We will now estimate $| \varepsilon^\circ |_s$ in terms of p_0 and p. We let $u(t, r, \varphi)$ denote the radial and $v(t, r, \varphi)$ the arc displacements of points of the plate. We multiply the first equation in (4.1) by u, the second by v, add and integrate over the domain Ω. Integrating by parts taking (3.3) into account, we will have

$$(\sigma^\circ, \varepsilon^\circ) \equiv \iint (\sigma_r^\circ \varepsilon_r^\circ + 2\sigma_{r\varphi}^\circ \varepsilon_{r\varphi}^\circ + \sigma_\varphi^\circ \varepsilon_\varphi^\circ)\, d\omega = J_2$$
$$(\varepsilon_r^\circ = \delta_r u, \; 2\varepsilon_{r\varphi}^\circ = r^{-1}\delta_\varphi u + \delta_r v - r^{-1} v, \; \varepsilon_\varphi^\circ = r^{-1} u + r^{-1}\delta_\varphi v)$$
$$J_2 = -\int_0^{2\pi} [pRu(R) - p_0 R_0 u(R_0)]\, d\varphi$$

Replacing the stress tensor components here by formjlas (2.1), we obtain

$$J^2 \equiv (1 - \nu) \iint (\varepsilon_r^{\circ 2} + 2\varepsilon_{r\varphi}^{\circ 2} + \varepsilon_\varphi^{\circ 2})\, d\omega + \nu \iint (\varepsilon_r^\circ + \varepsilon_\varphi^\circ)^2\, d\omega = \tag{4.3}$$

$$(1-\nu)\iint(\varepsilon_r^\circ N\varepsilon_r^\circ + 2\varepsilon_{r\varphi}^\circ N\varepsilon_{r\varphi}^\circ + \varepsilon_\varphi^\circ N\varepsilon_\varphi^\circ)\,d\omega + \nu\iint(\varepsilon_r^\circ+\varepsilon_\varphi^\circ)N(\varepsilon_r^\circ+\varepsilon_\varphi^\circ)\,d\omega + (1-\nu^2)E^{-1}J_2$$

We convert the quantity J_2 as follows:

$$J_2 = -\int_0^{2\pi}\Big[\Big(p\int_0^R - p_0\int_0^{R_0}\Big)(\delta_r u + r^{-1}u)\,r\,dr\Big]d\varphi =$$

$$-\int_0^{2\pi}\Big\{\Big[p\int_{R_0}^R + (p-p_0)\int_0^{R_0}\Big](\delta_r u + r^{-1}u + r^{-1}\delta_\varphi v)\,r\,dr\Big\}d\varphi =$$

$$-p\iint(\varepsilon_r^\circ+\varepsilon_\varphi^\circ)\,d\omega - (p-p_0)\int_0^{2\pi}\int_0^{R_0}(\varepsilon_r^\circ+\varepsilon_\varphi^\circ)\,r\,dr\,d\varphi$$

Then according to the Cauchy-Bunyakovskii inequality, we can obtain from (4.3) that

$$J^2 \leqslant J\int_0^t N_1(t,\tau)J\,d\tau + E^{-1}(1-\nu^2)(2\pi)^{1/2}\zeta\,\|\varepsilon^\circ\|$$

$$\zeta = p(R^2-R_0^2)^{1/2} + |p-p_0|R_0$$

Taking account of the inequality $J^2 \geqslant (1-\nu)\|\varepsilon^\circ\|^2$ we will hence have

$$(1-|N_1|)\,|\varepsilon^\circ|_s \leqslant (1+\nu)E^{-1}(2\pi)^{1/2}\zeta$$

Together with (4.2) this yields the estimate

$$|\sigma^\circ|_s \leqslant P = (2\pi)^{1/2}(1+\nu)(1+|N_1|)[(1-\nu)(1-|N_1|)]^{-1}\zeta \tag{4.4}$$

Using (4.4) to estimate I_3 from (3.5) and the result of the theorem, we conclude that the plate will be stable under the following condition on the compressive load:

$$P \leqslant D\lambda[h(1+|L_0|)]^{-1} \tag{4.5}$$

Remarks. 1°. If the plate material possesses just the properties 1°–2°, then conditions (3.11) and (4.5) take the respective form

$$\|\sigma^\circ\| \leqslant |\sigma^\circ|_s \leqslant D\lambda h^{-1}(1-|N_1|),\quad P \leqslant D\lambda h^{-1}(1-|N_1|)$$

To prove this fact (2.6) is used.

2°. The stability conditions (3.11) and (4.5) obtained retain their form even for other kinds of plate support. Only the parameter λ is found from (3.6) with boundary conditions corresponding to the kind of support.

3°. We assume that the plate is loaded in such a manner that the stresses within it are constant /2, 3/. Then the stability conditions are simplified and have the following form. We let $\lambda_1-\lambda_4$ denote the minimal eigenvalues of the boundary value problems

$$\Delta\Delta w + \lambda\Delta w = 0,\quad \Delta\Delta w + \lambda\delta_{rr}^2 w = 0$$

$$\Delta\Delta w + \lambda(r^{-1}\delta_r w + r^{-2}\delta_{\varphi\varphi}^2 w) = 0,\quad \Delta\Delta w + 2\lambda\delta_r(r^{-1}\delta_\varphi w) = 0$$

with boundary conditions corresponding to the kind of support. If $\sigma_r^\circ = \sigma_\varphi^\circ,\ \sigma_{r\varphi}^\circ = 0$, the plate is stable for $|\sigma_r^\circ| \leqslant f(\lambda_1)$; if $\sigma_\varphi^\circ = \sigma_{r\varphi}^\circ = 0$, it is stable for $|\sigma_r^\circ| \leqslant f(\lambda_2)$; if $\sigma_r^\circ = \sigma_{r\varphi}^\circ = 0$, it is stable for $|\sigma_\varphi^\circ| \leqslant f(\lambda_3)$; if $\sigma_r^\circ = \sigma_\varphi^\circ = 0$, it is stable for $|\sigma_{r\varphi}^\circ| < f(\lambda_4)$. Here $f(\lambda) = D\lambda[h(1+|L_0|)]^{-1}$.

For $|L_0| \equiv 0$ and $p_0 = p$ the conditions obtained agree with the stability conditions for elastic annular plates /2/.

REFERENCES

1. ARUTYUNYAN N.KH. and KOLMANOVSKII V.B., Creep Theory of Inhomogeneous Bodies, Nauka, Moscow, 1983.
2. VOL'MIR A.S., Stability of Deformable Sysyems. Nauka, Moscow, 1967.
3. TIMOSHENKO S.P. Stability of Elastic Systems /Russian translation/, Gostekhizdat, Moscow, 1955.
4. NOVOZHILOV V.V., Principles of the Non-linear Theory of Elasticity, Gostekhizdat, Moscow-Leningrad, 1948.
5. BRAYAN G.H., On the stability of a plane plate under thrusts in its own plane, with application to the "buckling" of the sides of a ship, Proc. London Math. Soc., Vol.22, 1891.
6. KOLMOGOROV A.N. and FOMIN S.V., Elements of Function Theory and Functional Analysis, Nauka, Moscow, 1976.
7. KOSHELEV A.I., A priori estimates in L_p and generalized solutions of elliptic equations and systems, Uspekhi Matematicheskikh Nauk, Vol.13, No.4, 1958.

8. LADYZHENSKAYA O.A., Boundary Value Problems of Mathematical Physics. Nauka, Moscow, 1973.
9. MIKHLIN S.G., Variational Methods in Mathematical Physics. Nauka, Moscow, 1970.

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POST-CRITICAL BEHAVIOUR OF A LONGITUDINALLY COMPRESSED ROD FOR RIGID LIMITATIONS ON THE DEFLECTION*

E.I. MIKHAILOVSKII, V.N. TARASOV and D.V. KHOLMOGOROV

An approach based on the application of optimization methods is developed for determining the state of stress and strain of bodies and structures with given limitation on the displacement. A model problem of plane longitudinal bending of a hinge-supported rod is considered with rigid limitations on the deflection. An analytic solution is obtained for this problem that extends a well-known solution to the non-linear case /1/. Then, by applyiny the Ritz method to a variational problem and replacing the continuous by discrete limitations, the variational problem is transformed into a non-linear programming problem. The results of numerical computations are in good agreement with the analytical solution. A simple proof is given for the complete adjacency hypothesis used to obtain the latter. The mechanism of the formation of the multiwave bending mode as the axial compressive force increases, described in /1/, is confirmed by a numerical experiment.

The problem under consideration is interesting in connection with the need to reveal the stable dynamic bending modes of drilling tube columns in a borehole. One of the methods of solving this problem is based on assumptions about the nature of adjacency of the column to the borehole wall or about the column bending mode. An investigation of the shape of a cambered axis using assumptions of complete adjacency is made in /2/.

We consider the plane bending of a longitudinally compressed rod located initially along the axis of a cylindrical cavity (the radius is $\Delta = \text{const}$) with absolutely rigid walls. Let the hinge-clamped ends of the rod remain on the cavity axis during deformation while the longitudinal compressive force P retains its magnitude and direction. Under such assumptions, the determination of the plane bending mode of the rod reduces to solving the following variational problem

$$\Pi[w] = \int_0^l L(w', w'')\, ds \to \min_{\|w\| \leqslant \Delta} \tag{1}$$
$$w(0) = w(l) = 0,\ w''(0) = w''(l) = 0$$
$$\left(L(w', w'') = \frac{EI}{2}\frac{w''^2}{1 - w'^2} - P\left(1 - \sqrt{1 - w'^2}\right)\right)$$

where w, w' and w'' are the deflection function, and its first and second derivatives with respect to s, EI is the rod bending stiffness, and l is the rod length.

Furthermore, we assume the force P to be greater than the first critical force $(P > P_*^{(1)} \equiv \pi^2 EI/l^2)$ and greater than the force for which the rod would touch the wall. We assume here that the rod abuts completely on a cavity wall at a certain middle part of length $l_2 = l - 2l_1$ (Fig.1). We call this assumption the hypothesis of total adjacency. When there is a section of total rectification, the determination of the deflection at each of the curvilinear sections (from the hinged end to the first point of tangency) reduces to solving the variational problem

$$\int_0^{l_1} L(w', w'')\, ds \to \min_{w,\ l_1} \tag{2}$$

under the boundary conditions

$$w(0) = w''(0) = 0,\ w(l_1) = \Delta,\ w'(l_1) = 0 \tag{3}$$

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